JOURNAL OF APPROXIMATION THEORY 14, 160–161 (1975)

Letter to the Editor

Nonexistence of Best Rational Approximations on Subsets

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario, London, Canada

Communicated by E. W. Cheney

Let Y be a compact subset of $[0, 1]$ and $C(Y)$ the space of continuous functions on Y. For $g \in C(Y)$ define

$$\| g \|_Y = \sup\{| g(x) | : x \in Y\}, \qquad \| g \| = \| g \|_{[0,1]} .$$

Let $R_m^n[0, 1]$ be the family of ratios p/q, p a polynomial of degree $\leqslant n$, q a polynomial of degree $\leqslant m$, $q(x) > 0$ for $0 \leqslant x \leqslant 1$. The Chebyshev approximation problem on Y is for given $f \in C(Y)$ to find an element r^* of $R_m^n[0, 1]$ such that $\| f - r \|_Y$ is minimized. Such an element r^* is called best to f on Y.

DEFINITION. An element r of $R_m^n[0, 1]$ is called *degenerate* if it can be expressed as p/q, p of degree $< n$, q of degree $< m$.

In [2] it is shown that if the best approximation to f on $[0, 1]$ is nondegenerate, then best approximations exist to f on all sufficiently dense subsets Y. We now consider the case where the best approximation r^* to f on $[0, 1]$ is degenerate. A point x is called an *extremum* of $f - r^*$ if

$$| f(x) - r^*(x) | = \| f - r^* \| .$$

If the set of extrema of $f - r^*$ is a union of closed intervals of positive length, it is readily seen that all sufficiently dense subsets Y of $[0, 1]$ contain a set of (alternating) extrema of $f - r^*$ and so r^* is best on Y. We consider the more typical case where endpoints are extrema and the extrema are isolated.

DEFINITION. Let $X_k \subset [0, 1]$. We say $\{X_k\} \to [0, 1]$ if for any $x \in [0, 1]$ there is a sequence $\{x_k\} \to x$, $x_k \in X_k$.

THEOREM. *Let r^* be a degenerate element of $R_m^n[0, 1]$. Let the set of extrema of $f - r^*$ be nowhere dense and an endpoint be an extremum. Then* *there exists a sequence* $\{X_k\} \to [0, 1]$ *such that no best approximation exists to* f *on* X_k.

Proof. Assume without loss of generality that 0 is an extremum and $f(0) - r^*(0) > 0$. Let $e = f(0) - r^*(0) = \|f - r^*\|$,

$$X_k = \{0\} \cup \{x : |f(x) - r^*(x)| \leqslant e - 1/k\}, \qquad 1/k < e.$$

Let r^* be represented by p^*/q^*, p^* of degree $n - 1$; q^* of degree $m - 1$, $q^*(x) > 0$ for $0 \leqslant x \leqslant 1$, $q^*(0) = 1$. Define

$$r_j(x) = r^*(x) + [e/j]/[x + 1/j]\, q^*(x).$$

As r^* is degenerate, $r_j \in R_m^n[0, 1]$.

Denote the norm on X_k by $\| \ \|_k$. We have $r_j(0) = f(0)$ and $r_j(x) \to r^*(x)$ uniformly for $x \in X_k \sim \{0\}$, hence $\{\|f - r_j\|_k \to e - 1/k$. Let r^* have alternating degree l then there exist $x_1^k < \cdots < x_l^k \in X_k$ on which $f - r^*$ attains alternately $-e + 1/k$ and $e - 1/k$. Let $x_0 = 0$. By the generalized lemma of de la Vallée-Poussin [1, p. 226], we have for $r \in R_m^n[0, 1] \sim r^*$,

$$\max\{|f(x_i^k) - r(x_i^k)| : i = 0,..., l\} > \min\{|f(x_i^k) - r^*(x_i^k)| : i = 0,..., l\} = e - 1/k.$$

Since $f(0) - r^*(0) = e$, there is no $r \in R_m^n[0, 1]$ with $\|f - r\|_k = e - 1/k$, that is, no best approximation exists on X_k.

The set X_k of the theorem is infinite. We are also interested in finite sets with similar properties. If we let

$$Y_k = [X_k \cap \{m/2^k\}] \cup \left[\bigcup_{j=1}^{k} \{x_1^j,..., x_l^j\}\right],$$

then Y_k is a finite set, $Y_k \subset Y_{k+1}$, $\{Y_k\} \to [0, 1]$, and best approximations do not exist on Y_k by identical arguments.

It should be noted that the results (and proof) of this paper can be extended to approximation by other alternating families, in particular to exponential sums.

References

1. C. B. Dunham, Chebyshev approximation with respect to a weight function, *J. Approximation Theory* **2** (1969), 223–232.
2. C. B. Dunham, Varisolvent chebyshev approximation on subsets, *in* "Approximation Theory" (G. G. Lorentz, ed.), pp. 337–340, Academic Press, New York, 1973.

Printed in Belgium